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Nambu Mechanics in the Lagrangian Formalism

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A Lagrangian formulation is presented as the counterpart of the Hamiltonian one for Nambu mechanics which is a natural generalization of Hamiltonian mechanics. If we postulate the existence of plural Lagrangians corresponding to the existence of plural Hamiltonians, we can formulate the Lagrangian formalism in Nambu mechanics as well as in Hamiltonian mechanics. Here, in terms of exterior differentiation, Nambu mechanics can be formulated in a completely parallel way to ordinary analytical mechanics, including generalized Legendre transformations.

1. INTRODUCTION

In 1973 Nambu proposed a generalization of ordinary Hamiltonian mechanics [1] which is now called *Nambu mechanics*. Many authors have since investigated its connection to the usual Hamiltonian mechanics and its quantization.

In this paper we study Nambu mechanics in terms of the exterior differential form and show that there exist plural Lagrangians corresponding to the number of Hamiltonians in it.

In Section 2 we explain Nambu mechanics briefly. Here a Nambu bracket, which is a generalization of the usual Poisson bracket, is explained as a natural generalization of the 19th century Jacobian form of the Poisson bracket [2].

In Section 3 we outline the canonical formulation of the mechanics by use of the Poincaré–Cartan exterior differential form $\Omega^{(1)}$:

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$$\Omega^{(1)} = p_i\, dq^i - H\, dt$$

In Section 4 we introduce the analog of the same exterior differential form for Nambu mechanics: for the three-dimensional case, it is

$$\Omega^{(2)} = q\, dp \wedge dr - H_1\, dH_2 \wedge dt$$

This $\Omega^{(2)}$ will be used for the principle of least action and "canonical" transformations in Nambu mechanics.

In Section 5 we formulate Nambu mechanics in a "Lagrangian" formalism. It is shown that we have two or more Lagrangians corresponding to the existence of two or more Hamiltonians in Nambu's formalism. In the case of two Lagrangians, they are connected with the corresponding Hamiltonians as

$$dH_1 \wedge dH_2 = \frac{1}{\dot{q}}\, d(p\dot{q} - L_1) \wedge d(r\dot{q} - L_2)$$

Here we see a generalized Legendre transformation in Nambu mechanics, as will be explained later.

2. NAMBU MECHANICS

In standard analytical mechanics, the equations of motion are

$$\begin{cases} \dot{q} = \dfrac{\partial H}{\partial p} \\[2mm] \dot{p} = -\dfrac{\partial H}{\partial q} \end{cases}$$

where these equations have an apparent asymmetry between q and p (q and p denote a doublet of dynamical variables, i.e., canonical coordinates and momenta, respectively, and H is the Hamiltonian of the system under consideration). The temporal development of any function $f(q, p)$ can be written in terms of the Poisson bracket as

$$\frac{df}{dt} = \{f, H\} \tag{1}$$

The Poisson bracket $\{.,.\}$ can also be rewritten in terms of the Jacobian,

$$\{f, H\} = \frac{\partial(f, H)}{\partial(q, p)}$$

as was often done in the l9th century [2]. Here we omitted the symbol of summation for the case of plural degrees of freedom. Noticing this form, Nambu [1] generalized formally the equations of motion (1) to

$$\frac{df}{dt} = \{f, H_1, H_2\} \equiv \frac{\partial(f, H_1, H_2)}{\partial(q, p, r)} \tag{2}$$

where q, p, and r denote a *triplet* of dynamical variables and H_1, H_2 are two Hamiltonians with arguments q, p, and r. Equation (2) is a natural generalization of Eq. (1) and the extended Poisson bracket $\{., ., .\}$ is called the *Nambu bracket*.

Now let us study the difference between the usual Hamiltonian mechanics and Nambu mechanics. In the usual Hamiltonian mechanics dynamical variables q, p are canonical pairs, hence the phase space spanned by (q, p) has even dimensions. In Nambu mechanics, however, the phase space has three or more (generally, any number of) dimensions, since Eq. (2) can be extended to the following:

$$\frac{df}{dt} = \frac{\partial(f, H_1, H_2, \ldots, H_{n-1})}{\partial(x_1, x_2, x_3, \ldots, x_n)} \tag{3}$$

where $(x_1, \ldots, x_n)$ denotes an *n-tuple* of dynamical variables and $H_1, \ldots, H_{n-1}$ are $n - 1$ Hamiltonians. It is shown by substituting H_k for f in Eq. (3) that each H_k is a constant of motion [or first integral of Eq. (3)]:

$$\begin{aligned}\frac{dH_k}{dt} &= \frac{\partial(H_k, H_1, \ldots, H_k, \ldots, H_{k-1})}{\partial(x_1, x_2, \ldots, x_n)} \\ &= 0\end{aligned}$$

because the Jacobian matrix has two equal rows. From this, we realize that in Nambu mechanics there are two or more Hamiltonians all of which are constants of motion, while the usual Hamiltonian mechanics involves a single Hamiltonian.

Let us return to the simplest triplet case. Substituting the triplet (q, p, r) for f respectively in Eq. (2), we have

$$\begin{cases} \dot{q} = \dfrac{\partial(q, H_1, H_2)}{\partial(q, p, r)} = \dfrac{\partial(H_1, H_2)}{\partial(p, r)} \\ \dot{p} = \dfrac{\partial(p, H_1, H_2)}{\partial(q, p, r)} = \dfrac{\partial(H_1, H_2)}{\partial(r, q)} \\ \dot{r} = \dfrac{\partial(r, H_1, H_2)}{\partial(q, p, r)} = \dfrac{\partial(H_1, H_2)}{\partial(q, p)} \end{cases} \tag{4}$$

We see hence the apparent asymmetry disappear, which we see in the usual Hamiltonian equations of motion.

Then, we can prove easily

$$\frac{\partial}{\partial q}\dot{q} + \frac{\partial}{\partial p}\dot{p} + \frac{\partial}{\partial r}\dot{r} = \frac{\partial}{\partial q}\frac{\partial(H_1, H_2)}{\partial(p, r)} + \frac{\partial}{\partial p}\frac{\partial(H_1, H_2)}{\partial(r, q)} + \frac{\partial}{\partial r}\frac{\partial(H_1, H_2)}{\partial(q, p)}$$
$$= 0$$

This indicates that the "divergence" of the velocity $(\dot{q}, \dot{p}, \dot{r})$ in a three-dimensional phase space vanishes, ensuring that the Liouville theorem is valid in Nambu mechanics as well as Hamiltonian mechanics. Since the Liouville theorem states that for an ensemble of identical systems the volume of the phase space occupied by the ensemble is conserved, we mention that the ensemble in the Nambu phase space is also supposed to be an incompressible fulid.

3. HAMILTONIAN MECHANICS IN TERMS OF THE DIFFERENTIAL FORM

In this section we outline Hamiltonian mechanics in terms of the exterior differential form as preparation for treating Nambu mechanics by it in the next section.

Let us begin by considering the following 1-form $\Omega^{(1)}$ on $\mathbb{R}^{2N+1}$ with N-dimensional coordinates $q^1, \ldots, q^N$, momenta $p_1, \ldots, p_N$, and a time t:

$$\Omega^{(1)} = p_i\, dq^i - H(q, p)\, dt \qquad (i = 1, \ldots, N) \tag{5}$$

where the summation convention is used. The reason why the canonical momenta p_i are covariant components of a vector in the above equation comes from the fact that p_i is defined via the Lagrangian $L(q, \dot{q})$ as

$$p_i \equiv \frac{\partial L}{\partial \dot{q}^i}$$

The 1-form $\Omega^{(1)}$, which is called the *fundamental 1-form* (or *Poincaré–Cartan integral invariant*), plays an important role in Hamiltonian mechanics. In fact, the principle of least action is represented in the form that the integral of $\Omega^{(1)}$ along the actual curve has an extremal:

$$\begin{aligned}\delta \int \Omega^{(1)} &= \delta \int (p_i\, dq^i - H\, dt)\\ &= \delta \int (p_i \dot{q}^i - H)\, dt\\ &= \delta \int L\, dt = 0\end{aligned} \tag{6}$$

which is Hamilton's principle.

The exterior differential of $\Omega^{(1)}$ is

$$\begin{aligned} d\Omega^{(1)} &= dp_i \wedge dq^i - \left(\frac{\partial H}{\partial q^i}\, dq^i + \frac{\partial H}{\partial p_i}\, dp_i\right) \wedge dt \\ &= \left(dp_i + \frac{\partial H}{\partial q^i}\, dt\right) \wedge \left(dq^i - \frac{\partial H}{\partial p_i}\, dt\right) \\ &= \theta_i \wedge \rho^i \end{aligned}$$

where

$$\begin{cases} \theta_i \equiv dp_i + \dfrac{\partial H}{\partial q^i}\, dt \\ \rho^i \equiv dq^i - \dfrac{\partial H}{\partial p_i}\, dt \end{cases}$$

This indicates the close relation between the fundamental 1-form $\Omega^{(1)}$ and the Hamiltonian equations of motion, because the Pfaffian equations

$$\begin{cases} \theta_i \equiv dp_i + \dfrac{\partial H}{\partial q^i}\, dt = 0 \\ \rho^i \equiv dq^i - \dfrac{\partial H}{\partial p_i}\, dt = 0 \end{cases}$$

are exactly the Hamiltonian equations,

$$\begin{cases} \dot{p}_i = -\dfrac{\partial H}{\partial q^i} \\ \dot{q}^i = \dfrac{\partial H}{\partial p_i} \end{cases} \tag{7}$$

Our next step is to find out how canonical transformations are expressed in this form. Since canonical transformations are those which preserve the Hamiltonian equations (7), we see from the above fact that the 2-form $d\Omega^{(1)}$ consists of θ_i and p^i that canonical transformations are those which preserve the 2-form $d\Omega^{(1)}$. For example, consider a transformation $(q^i, p_i) \to (Q^i, P_i)$ and suppose that the fundamental 1-form of the new coordinates and momenta system is

$$\Omega^{(1)'} = P_i\, dQ^i - K\,(Q, P)\, dt$$

Then the canonical transformation requires that

$$d\Omega^{(1)} = d\Omega^{(1)'} \tag{8}$$

In other words [since $d(dW) = 0$]

$$\Omega^{(1)} = \Omega^{(1)'} + dW \tag{9}$$

where W, which should be considered as the generating function of this transformation, is an arbitrary function. Equation (9) can be written as follows:

$$p_i \dot{q}^i - H(q, p) = P_i \dot{Q}^i - K(Q, P) + \frac{dW}{dt} \tag{10}$$

Assuming that $W = W(q, Q, t)$, we have

$$\frac{dW}{dt} = \frac{\partial W}{\partial q^i} \dot{q}^i + \frac{\partial W}{\partial Q^i} \dot{Q}^i + \frac{\partial W}{\partial t}$$

Substituting this equation into Eq. (10) and considering q^i and Q^i as independent variables, we obtain

$$\begin{cases} p_i = \dfrac{\partial W}{\partial q^i} \\ P_i = -\dfrac{\partial W}{\partial Q^i} \\ K = H + \dfrac{\partial W}{\partial t} \end{cases} \tag{11}$$

These equations show the relation between (q, p, H) and (Q, P, K).

In particular, provided that $W = W(q, Q)$,

$$\begin{aligned} dW &= \frac{\partial W}{\partial q^i} dq^i + \frac{\partial W}{\partial Q^i} dQ^i \\ &= p_i \, dq^i - P_i \, dQ^i \end{aligned}$$

where we employ Eq. (11). Taking the exterior differential of the above equation, we have

$$d\omega^{(1)} \equiv dq^i \wedge dp_i = dQ^i \wedge dP_i \equiv d\omega^{(1)'} \tag{12}$$

This equation shows the invariance of the 2-form $d\omega^{(1)}$ (which is called the *symplectic form*) under the canonical transformation. Equation (12) is equivalent to Eq. (8) in case $dt = 0$.

We see of course that the expression

$$\begin{aligned} (d\omega^{(1)})^N &= d\omega^{(1)} \wedge \ldots \wedge d\omega^{(1)} \\ &= (-1)^{N(N-1)/2} N! dq^1 \wedge \ldots \wedge dq^N \wedge dp_1 \wedge \ldots \wedge dp_N \\ &= (-1)^{N(N-1)/2} N! dV \end{aligned}$$

is also invariant according to Eq. (12); consequently, the volume element dV

on the phase space is preserved under the canonical transformation (this result corresponds to the Liouville theorem).

4. NAMBU MECHANICS IN TERMS OF THE DIFFERENTIAL FORM

Now we extend the result of Section 3 to Nambu mechanics. For simplicity, we will restrict ourselves to the three-dimensional case for a while.

The fundamental 1-form [namely, Eq. (5)] in Hamiltonian mechanics can be generalized to the following 2-form $\Omega^{(2)}$ on $\mathbb{R}^4$ of dynamical variables q, p, r, and a time t [3–5]:

$$\Omega^{(2)} = q\, dp \wedge dr - H_1\, dH_2 \wedge dt \tag{13}$$

The reason why we choose the above 2-form in Nambu mechanics is shown as follows: The differential of $\Omega^{(2)}$ is written as

$$\begin{aligned} d\Omega^{(2)} &= dq \wedge dp \wedge dr - \left(\frac{\partial H_1}{\partial q}\, dq + \frac{\partial H_1}{\partial p}\, dp + \frac{\partial H_1}{\partial r}\, dr\right) \\ &\quad \wedge \left(\frac{\partial H_2}{\partial q}\, dq + \frac{\partial H_2}{\partial p}\, dp + \frac{\partial H_2}{\partial r}\, dr\right) \wedge dt \\ &= \left(dq - \frac{\partial(H_1, H_2)}{\partial(p, r)}\, dt\right) \wedge \left(dp - \frac{\partial(H_1, H_2)}{\partial(r, q)}\, dt\right) \\ &\quad \wedge \left(dr - \frac{\partial(H_1, H_2)}{\partial(q, p)}\, dt\right) \\ &= \theta \wedge \rho \wedge \sigma \end{aligned}$$

where

$$\begin{cases} \theta = dq - \dfrac{\partial(H_1, H_2)}{\partial(p, r)}\, dt \\ \rho = dp - \dfrac{\partial(H_1, H_2)}{\partial(r, q)}\, dt \\ \sigma = dr - \dfrac{\partial(H_1, H_2)}{\partial(q, p)}\, dt \end{cases}$$

Now the Pfaffian equations,

$$\begin{cases} \theta = dq - \dfrac{\partial(H_1, H_2)}{\partial(p, r)}\, dt = 0 \\ \rho = dp - \dfrac{\partial(H_1, H_2)}{\partial(r, q)}\, dt = 0 \\ \sigma = dr - \dfrac{\partial(H_1, H_2)}{\partial(q, p)}\, dt = 0 \end{cases}$$

are equivalent to Eqs. (4). Therefore, by analogy with the Hamiltonian case in Section 3, Eq. (13) can be considered as the generalized fundamental 2-form in Nambu mechanics, which corresponds to the 1-form in Hamiltonian mechanics.

In the following we investigate canonical transformations in Nambu mechanics. In the same way as for Hamiltonian mechanics, we call a mapping g: $(q, p, r) \rightarrow (Q, P, R)$ the canonical transformation in Nambu mechanics if g preserves the 3-form $d\omega^{(2)} = dq \wedge dp \wedge dr$ [which corresponds to the generalized one of Eq. (12)]:

$$d\omega^{(2)} = dq \wedge dp \wedge dr = dQ \wedge dP \wedge dR = d\omega^{(2)'}$$

Then

$$\begin{aligned} dq \wedge dp \wedge dr &= \left(\frac{\partial Q}{\partial q}\, dq + \frac{\partial Q}{\partial p}\, dp + \frac{\partial Q}{\partial r}\, dr\right) \\ &\quad \wedge \left(\frac{\partial P}{\partial q}\, dq + \frac{\partial P}{\partial p}\, dp + \frac{\partial P}{\partial r}\, dr\right) \wedge \left(\frac{\partial R}{\partial q}\, dq + \frac{\partial R}{\partial p}\, dp + \frac{\partial R}{\partial r}\, dr\right) \\ &= \frac{\partial(Q, P, R)}{\partial(q, p, r)}\, dq \wedge dp \wedge dr \end{aligned}$$

Consequently

$$\frac{\partial(Q, P, R)}{\partial(q, p, r)} = 1$$

By virtue of this, Eq. (2) is shown to be invariant under the canonical transformation:

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial(f, H_1, H_2)}{\partial(q, p, r)} \\ &= \frac{\partial(f, H_1, H_2)}{\partial(Q, P, R)} \frac{\partial(Q, P, R)}{\partial(q, p, r)} \\ &= \frac{\partial(f, H_1, H_2)}{\partial(Q, P, R)} \end{aligned}$$

The 2-form (13) can be easily generalized to the $(n-1)$-form on $\mathbb{R}^{n+1}$ with $x_1, \ldots, x_n, t$;

$$\Omega^{(n-1)} = x_1 dx_2 \wedge \cdots \wedge dx_n - H_1\, dH_2 \wedge \cdots \wedge dH_{n-1} \wedge dt$$

which is the invariant form corresponding to the case of $n-1$ Hamiltonians, n being any integer more than 3. In this case the same arguments as above still hold.

5. EXISTENCE OF TWO OR MORE LAGRANGIANS IN NAMBU MECHANICS

Here we consider Lagrangians in Nambu mechanics which have not been sufficiently considered by other researchers. In what follows, we show there exist *two or more* Lagrangians corresponding to two or more Hamiltonians in Nambu mechanics.

Before presenting plural Lagrangians, we comment on the single Lagrangian proposed by Bayen and Flato [6]. Their Lagrangian is

$$L(\vec{x}, \dot{\vec{x}}) = H_1(\vec{x})\dot{\vec{x}} \cdot \vec{\nabla} H_2(\vec{x}) \tag{14}$$

($\vec{x}$ being the configuration variables). Remark that this Lagrangian is linear in velocities. The Euler–Lagrange equation for this Lagrangian is

$$\dot{\vec{x}} \times (\vec{\nabla} H_1 \times \vec{\nabla} H_2) = 0$$

from which we deduce

$$\dot{\vec{x}} = f(\vec{x})(\vec{\nabla} H_1 \times \vec{\nabla} H_2)$$

with an arbitrary function $f(\vec{x})$. In the case $f(\vec{x}) = 1$,

$$\begin{aligned} \dot{\vec{x}} &= \vec{\nabla} H_1 \times \vec{\nabla} H_2 \\ &= \left(\frac{\partial(H_1, H_2)}{\partial(y, z)}, \frac{\partial(H_1, H_2)}{\partial(z, x)}, \frac{\partial(H_1, H_2)}{\partial(x, y)} \right) \end{aligned} \tag{15}$$

where $\vec{x} = (x, y, z)$. Indeed, these equations are formally equivalent to Eqs. (4), but do not well fit the statement that Nambu mechanics is a natural generalization of Hamiltonian mechanics (recall Section 2). In Eq. (15) the coordinates $\vec{x} = (x, y, z)$ denote the configuration variables on the three-dimensional space, whereas in the Nambu equations of motion (4) only q denotes the configuration variable. The variables p and r are, so to speak, the *first* canonical momentum and the *second* canonical momentum, respectively, for the coordinate q. Moreover the canonical momenta derived in the usual way from their Lagrangian (14), $\partial L/\partial\dot{\vec{x}} = H_1\vec{\nabla} H_2$, play no role in their example. Thus Eq. (15) is different from the Nambu equations of motion; the Lagrangian in the form (14) is irrelevant to Nambu mechanics.

Another Lagrangian that may be considered by examining the principle of least action in Nambu mechanics was proposed by Takhtajan [5]. Corresponding to the 1-form $\Omega^{(1)}$ of the usual action integral (6) in Hamiltonian mechanics, we consider the integral of the 2-form $\Omega^{(2)}$ (13) as Takhtajan did:

$$\iint \Omega^{(2)} = \iint (q\, dp\, dr - H_1\, dH_2\, dt)$$

(we omit the notation $\wedge$ from now on). He assumed that q, p, r are functions of two parameters t and t' corresponding to the double integration, i.e., $q = q(t, t')$, $p = p(t, t')$, $r = r(t, t')$, and that variations δq, δp, δr are zero at the endpoints of each parameter. Then,

$$\begin{aligned}
\iint \Omega^{(2)} = \iint \Bigg[& q\left(\frac{\partial p}{\partial t}\, dt + \frac{\partial p}{\partial t'}\, dt'\right)\left(\frac{\partial r}{\partial t}\, dt + \frac{\partial r}{\partial t'}\, dt'\right) \\
& - H_1 \left\{ \frac{\partial H_2}{\partial q}\left(\frac{\partial q}{\partial t}\, dt + \frac{\partial q}{\partial t'}\, dt'\right)\right. \\
& \left. + \frac{\partial H_2}{\partial p}\left(\frac{\partial p}{\partial t}\, dt + \frac{\partial p}{\partial t'}\, dt'\right) + \frac{\partial H_2}{\partial r}\left(\frac{\partial r}{\partial t}\, dt + \frac{\partial r}{\partial t'}\, dt'\right)\right\} dt \Bigg] \\
= \iint \Bigg[& q\left(\frac{\partial p}{\partial t'}\frac{\partial r}{\partial t} - \frac{\partial p}{\partial t}\frac{\partial r}{\partial t'}\right) \\
& - H_1\left(\frac{\partial H_2}{\partial q}\frac{\partial q}{\partial t'} + \frac{\partial H_2}{\partial p}\frac{\partial p}{\partial t'} + \frac{\partial H_2}{\partial r}\frac{\partial r}{\partial t'}\right)\Bigg] dt'\, dt
\end{aligned} \tag{16}$$

Thus,

$$\begin{aligned}
\delta \iint \Omega^{(2)} = \iint \Bigg[& \delta q\left(\frac{\partial p}{\partial t'}\frac{\partial r}{\partial t} - \frac{\partial p}{\partial t}\frac{\partial r}{\partial t'}\right) + q\delta\left(\frac{\partial p}{\partial t'}\frac{\partial r}{\partial t} - \frac{\partial p}{\partial t}\frac{\partial r}{\partial t'}\right) \\
& - \delta H_1\left(\frac{\partial H_2}{\partial q}\frac{\partial q}{\partial t'} + \frac{\partial H_2}{\partial p}\frac{\partial p}{\partial t'} + \frac{\partial H_2}{\partial r}\frac{\partial r}{\partial t'}\right) \\
& - H_1\delta\left(\frac{\partial H_2}{\partial q}\frac{\partial q}{\partial t'} + \frac{\partial H_2}{\partial p}\frac{\partial p}{\partial t'} + \frac{\partial H_2}{\partial r}\frac{\partial r}{\partial t'}\right)\Bigg] dt'\, dt
\end{aligned}$$

In the above expression, a collection of all the terms which involve δq is

$$\iint \left[\left(\frac{\partial p}{\partial t'} \frac{\partial r}{\partial t} - \frac{\partial p}{\partial t} \frac{\partial r}{\partial t'} \right) \delta q \right.$$
$$- \frac{\partial H_1}{\partial q} \left(\frac{\partial H_2}{\partial q} \frac{\partial q}{\partial t'} + \frac{\partial H_2}{\partial p} \frac{\partial p}{\partial t'} + \frac{\partial H_2}{\partial r} \frac{\partial r}{\partial t'} \right) \delta q$$
$$\left. - H_1 \left(\frac{\partial q}{\partial t'} \frac{\partial^2 H_2}{\partial q^2} \delta q + \frac{\partial H_2}{\partial q} \frac{\partial}{\partial t'} \delta q + \frac{\partial p}{\partial t'} \frac{\partial^2 H_2}{\partial p \partial q} \delta q + \frac{\partial r}{\partial t'} \frac{\partial^2 H_2}{\partial r \partial q} \delta q \right) \right] dt' \, dt \tag{17}$$

The seventh term in Eq. (17) can be written, integrating by parts and using the previous assumption with respect to the endpoints of t', as follows:

$$- \iint H_1 \frac{\partial H_2}{\partial q} \frac{\partial}{\partial t'} (\delta q) \, dt' \, dt$$
$$= \iint \frac{\partial}{\partial t'} \left(H_1 \frac{\partial H_2}{\partial q} \right) \delta q \, dt' \, dt$$
$$= \iint \left[\frac{\partial}{\partial q} \left(H_1 \frac{\partial H_2}{\partial q} \right) \frac{\partial q}{\partial t'} + \frac{\partial}{\partial p} \left(H_1 \frac{\partial H_2}{\partial q} \right) \frac{\partial p}{\partial t'} \right.$$
$$\left. + \frac{\partial}{\partial r} \left(H_1 \frac{\partial H_2}{\partial q} \right) \frac{\partial r}{\partial t'} \right] \delta q \, dt' \, dt$$
$$= \iint \left[\frac{\partial H_1}{\partial q} \frac{\partial H_2}{\partial q} \frac{\partial q}{\partial t'} + H_1 \frac{\partial^2 H_2}{\partial q^2} \frac{\partial q}{\partial t'} + \frac{\partial H_1}{\partial p} \frac{\partial H_2}{\partial q} \frac{\partial p}{\partial t'} + H_1 \frac{\partial^2 H_2}{\partial p \partial q} \frac{\partial p}{\partial t'} \right.$$
$$\left. + \frac{\partial H_1}{\partial r} \frac{\partial H_2}{\partial q} \frac{\partial r}{\partial t'} + H_1 \frac{\partial^2 H_2}{\partial r \partial q} \frac{\partial r}{\partial t'} \right] \delta q \, dt' \, dt$$

Substituting this to Eq. (17), we see that Eq. (17) is equal to

$$\iint \left[\left(\frac{\partial r}{\partial t} - \frac{\partial (H_1, H_2)}{\partial (q, p)} \right) \frac{\partial p}{\partial t'} - \left(\frac{\partial p}{\partial t} - \frac{\partial (H_1, H_2)}{\partial (r, q)} \right) \frac{\partial r}{\partial t'} \right] \delta q \, dt' \, dt$$

Similar calculations for δp and δr give

$$\delta \iint \Omega^{(2)} = \iint \left[\left\{ \left(\frac{\partial r}{\partial t} - \frac{\partial (H_1, H_2)}{\partial (q, p)} \right) \frac{\partial p}{\partial t'} - \left(\frac{\partial p}{\partial t} - \frac{\partial (H_1, H_2)}{\partial (r, q)} \right) \frac{\partial r}{\partial t'} \right\} \delta q \right.$$
$$+ \left\{ \left(\frac{\partial q}{\partial t} - \frac{\partial (H_1, H_2)}{\partial (p, r)} \right) \frac{\partial r}{\partial t'} - \left(\frac{\partial r}{\partial t} - \frac{\partial (H_1, H_2)}{\partial (q, p)} \right) \frac{\partial q}{\partial t'} \right\} \delta p$$

$$+ \left\{ \left(\frac{\partial p}{\partial t} - \frac{\partial(H_1, H_2)}{\partial(r, q)} \right) \frac{\partial q}{\partial t'} - \left(\frac{\partial q}{\partial t} - \frac{\partial(H_1, H_2)}{\partial(p, r)} \right) \frac{\partial p}{\partial t'} \right\} \delta r \Bigg] \, dt' \, dt$$

This indicates that if Eqs. (4) are satisfied, $\delta \iint \Omega^{(2)} = 0$ for arbitrary variations δq, δp, δr. Therefore, it appears that the principle of least action is valid also for Nambu mechanics in the above form and we have the integrand under the action integral, which should be considered the Lagrangian function,

$$L(q, p, r) = \int \left[q \left(\frac{\partial p}{\partial t'} \frac{\partial r}{\partial t} - \frac{\partial p}{\partial t} \frac{\partial r}{\partial t'} \right) - H_1 \left(\frac{\partial H_2}{\partial q} \frac{\partial q}{\partial t'} + \frac{\partial H_2}{\partial p} \frac{\partial p}{\partial t'} + \frac{\partial H_2}{\partial r} \frac{\partial r}{\partial t'} \right) \right] dt'$$

from the analogy of Eq. (6), viewing the form of $\Omega^{(2)}$, (16). Though we see that the Nambu equations of motion are derived in this way using two parameters t and t' for the action integral, the physical meaning of the second paremeter t' is ambiguous; moreover, this Lagrangian is not a suitable one since the function does not consist of coordinates and their time derivatives. In addition, a single Lagrangian could not provide the Legendre transformation to generate two or more Hamiltonians.

Now, how can we obtain the genuine Lagrangians in Nambu mechanics? For this purpose, we postulate that *there exist as many Lagrangians as the number of Hamiltonians* in Nambu mechanics and that for each Lagrangian the principle of least action holds. In the simplest case we postulate the existence of *two* Lagrangians $L_1(q, \dot{q})$, $L_2(q, \dot{q})$ corresponding to the two Hamiltonians. That is,

$$\begin{cases} \delta \int L_1(q, \dot{q}) \, dt = 0 \\ \delta \int L_2(q, \dot{q}) \, dt = 0 \end{cases}$$

Thus we have the two Euler–Lagrange equations for the two Lagrangians:

$$\begin{cases} \dfrac{\partial L_1}{\partial q} - \dfrac{d}{dt} \left(\dfrac{\partial L_1}{\partial \dot{q}} \right) = 0 \\ \dfrac{\partial L_2}{\partial q} - \dfrac{d}{dt} \left(\dfrac{\partial L_2}{\partial \dot{q}} \right) = 0 \end{cases} \tag{18}$$

Next we define the *first* canonical momentum p and the *second* canonical momentum r as

$$\begin{cases} p \equiv \dfrac{\partial L_1}{\partial \dot{q}} \\ r \equiv \dfrac{\partial L_2}{\partial \dot{q}} \end{cases} \tag{19}$$

From Eqs. (18), we see that

$$\begin{cases} \dot{p} = \dfrac{\partial L_1}{\partial q} \\ \dot{r} = \dfrac{\partial L_2}{\partial q} \end{cases} \tag{20}$$

Now we define the two Hamiltonians H_1 and H_2 through exterior differentials as follows:

$$dH_1 \wedge dH_2 \equiv \frac{1}{\dot{q}}\, d(p\dot{q} - L_1) \wedge d(r\dot{q} - L_2) \tag{21}$$

where H_1 and H_2 are expressed in terms of q, p, and r. Then the left-hand side of Eq. (21) is equal to

$$\begin{aligned} dH_1 \wedge dH_2 &= \left(\frac{\partial H_1}{\partial q} dq + \frac{\partial H_1}{\partial p} dp + \frac{\partial H_1}{\partial r} dr\right) \wedge \left(\frac{\partial H_2}{\partial q} dq + \frac{\partial H_2}{\partial p} dp + \frac{\partial H_2}{\partial r} dr\right) \\ &= \frac{\partial(H_1, H_2)}{\partial(p, r)} dp \wedge dr + \frac{\partial(H_1, H_2)}{\partial(r, q)} dr \wedge dq + \frac{\partial(H_1, H_2)}{\partial(q, p)} dq \wedge dp \end{aligned}$$

and the right-hand side of Eq. (21) is equal to, by virtue of Eqs. (19) and Eqs. (20),

$$\begin{aligned} &\frac{1}{\dot{q}}\, d(p\dot{q} - L_1) \wedge d(r\dot{q} - L_2) \\ &\quad = \frac{1}{\dot{q}} \left[\dot{q}\, dp + p\, d\dot{q} - \left(\frac{\partial L_1}{\partial q} dq + \frac{\partial L_1}{\partial \dot{q}} d\dot{q}\right)\right] \\ &\qquad \wedge \left[\dot{q}\, dr + r\, d\dot{q} - \left(\frac{\partial L_2}{\partial q} dq + \frac{\partial L_2}{\partial \dot{q}} d\dot{q}\right)\right] \\ &\quad = \frac{1}{\dot{q}} (\dot{q}\, dp - \dot{p}\, dq) \wedge (\dot{q}\, dr - \dot{r}\, dq) \\ &\quad = \dot{q}\, dp \wedge dr + \dot{p}\, dr \wedge dq + \dot{r}\, dq \wedge dp \end{aligned}$$

Comparing these equations, we obtain

$$\begin{cases} \dot{q} = \dfrac{\partial(H_1, H_2)}{\partial(p, r)} \\ \dot{p} = \dfrac{\partial(H_1, H_2)}{\partial(r, q)} \\ \dot{r} = \dfrac{\partial(H_1, H_2)}{\partial(q, p)} \end{cases}$$

which are the same equations as Eqs. (4). This guarantees the reasonableness of the postulate of the existence of two Lagrangians and our definition of Hamiltonians corresponding to them in Eq. (21).

The generalization of the preceding results to the case including multiple *Nambu momenta*, where dynamical variables consist of n-tuples $(x_1, \ldots, x_n)$ $[= (q, p_1, \ldots, p_{n-1})]$, is straightforward. Remark that there exist $n - 1$ Lagrangians $L_1(x_1, \dot{x}_1), \ldots, L_{n-1}(x_1, \dot{x}_1)$ and the principle of least action holds for each L_k. The Nambu momenta are defined as follows. The *first* momentum $x_2 \equiv \partial L_2/\partial \dot{x}_1$ (i.e., $p_1 \equiv \partial L_1/\partial \dot{q}$), the *second* momentum $x_3 \equiv \partial L_2/\partial \dot{x}_1$ (i.e., $p_2 \equiv \partial L_2/\partial \dot{q}$), ..., the $(n - 1)$ th momentum $x_n \equiv \partial L_{n-1}/\partial \dot{x}_1$ (i.e., $p_{n-1} \equiv \partial L_{n-1}/\partial \dot{q}$). The $n - 1$ Hamiltonians $H_1(x_1, \ldots, x_n), \ldots, H_{n-1}(x_1, \ldots, x_n)$ are defined as

$$dH_1 \wedge \cdots \wedge dH_{n-1} \equiv (\dot{x}_1)^{-(n-2)}\, d(x_2\dot{x}_1 - L_1) \wedge \cdots \wedge d(x_n\dot{x}_1 - L_{n-1}) \qquad (n \geq 2) \tag{22}$$

Then we obtain

$$\begin{cases} \dot{x}_1 = \dfrac{\partial(H_1, \ldots, H_{n-1})}{\partial(x_2, \ldots, x_n)} \\ \quad\vdots \\ \dot{x}_k = (-1)^{k+1} \dfrac{\partial(H_1, \ldots, H_{n-1})}{\partial(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)} \\ \quad\vdots \\ \dot{x}_n = (-1)^{n+1} \dfrac{\partial(H_1, \ldots, H_{n-1})}{\partial(x_1, \ldots, x_{n-1})} \end{cases}$$

as the equations of motion.

Using these equations, we have for any function $f = f(x_1, \ldots, x_n)$

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial x_1}\dot{x}_1 + \cdots + \frac{\partial f}{\partial x_k}\dot{x}_k + \cdots + \frac{\partial f}{\partial x_n}\dot{x}_n \\ &= \frac{\partial f}{\partial x_1}\frac{\partial(H_1, \ldots, H_{n-1})}{\partial(x_2 \ldots, x_n)} + \cdots \end{aligned}$$

$$
\begin{aligned}
&+ \frac{\partial f}{\partial x_k}(-1)^{k+1}\frac{\partial(H_1, \ldots, H_{n-1})}{\partial(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)} \\
&+ \cdots + \frac{\partial f}{\partial x_n}(-1)^{n+1}\frac{\partial(H_1, \ldots, H_{n-1})}{\partial(x_1, \ldots, x_{n-1})} \\
= &\frac{\partial(f, H_1, \ldots, H_{n-1})}{\partial(x_1, x_2, \ldots, x_n)}
\end{aligned}
$$

This is just equal to the most generalized form of the Nambu equation of motion (3).

For $n = 2$ Eq. (22) turns out to be the differential form of the definition of the Hamiltonian in the Legendre transformation in standard analytical mechanics:

$$dH = d(p\dot{q} - L) \tag{23}$$

where we employ q and p instead of x_1 and x_2. Therefore we can conclude that Eq. (22), the definition of Hamiltonians in the case of many Hamiltonians via many Lagrangians, is a natural generalization of Eq. (23). We have shown the generalized Legendre transformation valid in the Nambu formalism of mechanics; it supplements the lack of a relation between Lagrangians and Hamiltonians in developments hitherto made for Nambu mechanics.

6. CONCLUSION

Nambu mechanics was proposed as a generalization of the usual Hamiltonian mechanics. In analytical mechanics, we have the Lagrangian form of mechanics in parallel to the Hamiltonian one, and there is a connection that each central function is transformed by the Legendre transformation together with the change of fundamental dynamical variables $(q, \dot{q}) \leftrightarrow (q, p)$. If Nambu mechanics is to be a genuine generalization of usual analytical mechanics, it should have its counterpart of Lagrangian form.

In papers about Nambu mechanics, many authors have laid emphasis on the Hamiltonian formalism and not on the Lagrangian formalism; the appropriate Lagrangian formalism for Nambu mechanics has not been given even if the Lagrangians were treated. In this paper (especially, in Section 5), by use of the exterior differential form, we formulated Nambu mechanics in the Lagrangian formalism and showed that Nambu mechanics is formulated as well in the Lagrangian formalism as in the Hamiltonian formalism. We found that the exterior differential form offers great advantages in formulating Nambu mechanics.

Some authors [6–11] concluded that Nambu mechanics could be embedded in Hamiltonian mechanics. As we have seen in this paper, however, Nambu mechanics, which was first proposed as a generalization of the usual Hamiltonian mechanics, can be formulated as well in the Lagrangian form. The meaning of the generalization is as follows. In the usual Lagrangian and Hamiltonian formalism the q's are generalized coordinates and the p's are their conjugate momenta defined as $p \equiv \partial L/\partial \dot{q}$, while in our formulation of Nambu mechanics the q's are the same as in the usual case, but the p's are two or more variables corresponding to $p_1 \equiv \partial L_1/\partial \dot{q}$, $p_2 \equiv \partial L_2/\partial \dot{q}, \ldots$. In the simplest case, for example, we may take the triplet of dynamical variables (q, p, r) as a coordinate (q) and the *first* and the *second* momentum (p, r). As a result, we can say, in contrast to refs. 6–11, that Hamilton mechanics is to be embedded in Nambu mechanics; namely, the usual Hamiltonian (and Lagrangian) mechanics is interpreted as a special case of Nambu mechanics in which the number of Hamiltonians (and consequently Lagrangians) is only one.

We expect to find that Nambu mechanics can be used to describe certain physical phenomena that cannot have be appropriately described by means of the usual Hamiltonian mechanics.

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